RATIONAL CURVES ON CALABI-YAU THREEFOLDS

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In the conformal field theory arising from the compactification of strings on a Calabi-Yau threefold X, there naturally arise fields correspond to harmonic forms of types (2,1) and (1,1) on X [4]. The uncorrected Yukawa couplings in $H^{2,1}$ and $H^{1,1}$ are cubic forms that can be constructed by techniques of algebraic geometry — [20] contains a nice survey of this in a general context in a language written for mathematicians. The cubic form on the space $H^{p,1}$ of harmonic $(p,1)$ forms is given by the intersection product $(\omega_i, \omega_j, \omega_k) \mapsto \int_X \omega_i \wedge \omega_j \wedge \omega_k$ for $p = 1$, while for $p = 2$ there is a natural formulation in terms of infinitesimal variation of Hodge structure [20, 6]. The Yukawa couplings on $H^{2,1}$ are exact, while those on $H^{1,1}$ receive instanton corrections. In this context, the *instantons* are non-constant holomorphic maps $f : \mathbb{CP}^1 \to X$. The image of such a map is a *rational curve* on X, which may or may not be smooth. If the rational curve C does not move inside X, then the contribution of the instantons which are generically 1-1 (i.e. birational) maps with image C can be written down explicitly — this contributions only depends on the topological type of C, or more or less equivalently, the integrals $\int_C f^* J$ for (1files, a,1) forms J on X.

If the conformal field theory could also be expressed in terms of a "mirror manifold" X', then the uncorrected Yukawa couplings on $H^{2,1}(X')$ would be the same as the corrected Yukawa couplings on $H^{1,1}(X)$. So if identifications could be made properly, the infinitesimal variation of Hodge structure on X' would give information on the rational curves on X. In a spectacular paper [5], Candelas et. al. do this when X is a quintic threefold. Calculating the Yukawa coupling on $H^{2,1}(X')$ as the complex structure of X' varies gives the Yukawa coupling on $H^{1,1}(X)$ as J varies. The coefficients of the resulting Fourier series are then directly related to the instanton corrections.

To the mathematician, there are some unanswered questions in deducing the number of rational curves of degree d on X from this [20]. I merely cite one problem here, while I will state a little differently. It is not yet known how to carry out the calculation of the instanton

correction associated to a continuous family of rational curves on X. If there were continuous families of rational curves on a general quintic threefold X, this would complicate the instanton corrections to the Yukawa coupling on the space of $(1,1)$ forms. The question of whether or not such families exist has not yet been resolved. But even worse, this complication is present in any case — any degree m mapping of $\mathbb{CP}^1$ to itself may be composed with any birational mapping of $\mathbb{CP}^1$ to X to give a new map from $\mathbb{CP}^1$ to X; and this family has has moduli than mere reparametrizations of the sphere. In [5] this was explicitly recognized; the assertion was made that such a family counts $1/m^3$ times. This has since been verified by Aspinwall and Morrison [3].

My motivation in writing this note is to give a general feel for the mathematical meaning of "the number of rational curves on a Calabi-Yau threefold", in particular, how to "count" a family of rational curves. I expect that these notions will directly correspond to the not as yet worked out procedure for calculating instanton corrections associated to general families of rational curves. More satisfactory mathematical formulations are the subject of work still in progress.

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1. Rational Curves, Normal Bundles, Deformations

Consider a Calabi-Yau threefold X containing a smooth rational curve
$C \cong \mathbb{CP}^1$. The normal bundle $N_{C/X}$ of C in X is defined by the exact sequence

$$(1) \qquad\qquad 0 \to T_C \to T_X|_C \to N_{C/X} \to 0.$$

$N = N_{C/X}$ is a rank 2 vector bundle on C, so $N = \mathcal{O}(a) \oplus \mathcal{O}(b)$ for some integers a, b. Now $c_1(T_C) = 2$, and $c_1(T_X) = 0$ by the Calabi-Yau condition. So the exact sequence (1) yields $c_1(N) = -2$, or $a + b = -2$. One "expects" $a = b = -1$ in the general case. This is because there is a moduli space of deformations of the vector bundle $\mathcal{O}(a) \oplus \mathcal{O}(b)$, and the general point of this moduli space is $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, no matter what a and b are, as long as $a + b = -2$.

Let $\mathcal{M}$ be the moduli space of rational curves in X. The tangent space to $\mathcal{M}$ at C is given by $H^0(N)$ [19, §12]. In other words, $\mathcal{M}$ may be locally defined by finitely many equations in $\dim H^0(N)$ variables.

Definition. C is *infinitesimally rigid* if $H^0(N) = 0$.

Infinitesimal rigidity means that C does not deform inside X, not even to first order. Note that $H^0(N) = 0$ if and only if $a = b = -1$. Thus

- C is infinitesimally rigid if and only if $a = b = -1$.
- C deforms, at least infinitesimally, if and only if $(a, b) \neq (-1, -1)$.

$\mathcal{M}$ can split up into countably many irreducible components. For instance, curves with distinct homology classes in X will lie in different components of $\mathcal{M}$. However, there will be at most finitely many components of $\mathcal{M}$ corresponding to rational curves in a fixed homology class.

There certainly exist Calabi-Yau threefolds X containing positive dimensional families of rational curves. For instance, the Fermat quintic threefold $x_0^5 + \ldots x_4^5 = 0$ contains the family of lines given parametrically in the homogeneous coordinates (u, v) of $\mathbb{P}^1$ by $(u, -u, av, bv, cv)$, where (a, b, c) are the parameters of the plane curve $a^5 + b^5 + c^5 = 0$. However, suppose that all rational curves on X have normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Then since $\mathcal{M}$ consists entirely of discrete points, the remarks above show that there would be only finitely many rational curves in X in a fixed homology class. Since a quintic threefold has $H_2(X, \mathbb{Z}) \cong \mathbb{Z}$, the degree of a curve is essentially the same as its homology class. This discussion leads to Clemens' conjecture [8]:

Conjecture (Clemens). *A general quintic threefold contains only finitely many rational curves of degree d, for any $d \in \mathbb{Z}$. These curves are all infinitesimally rigid.*

Clemens' original constant count went as follows [7]: a rational curve of degree d in $\mathbb{P}^4$ is given parametrically by 5 forms $\alpha_0(u, v), \ldots, \alpha_4(u, v)$, each homogeneous of degree d in the homogeneous coordinates $(u : v)$ of $\mathbb{P}^1$. These α_i depend on $5(d + 1)$ parameters. On the other hand, a quintic equation $F(x_0, \ldots, x_4) = 0$ imposes the condition $F(\alpha_0(u, v), \ldots, \alpha_4(u, v)) \equiv 0$ for the parametric curve to be contained in this quintic threefold. This is a polynomial equation of degree $5d$ in u and v. Since a general degree $5d$ polynomial $\sum a_i u^i v^{5d-i}$ has $5d+1$ coefficients, setting these equal to zero results in $5d + 1$ equations among the $5d + 5$ parameters of the α_i. If F is *general*, it seems plausible that these equations should impose independent conditions, so that the solutions should depend on $5d + 5 - (5d + 1) = 4$ parameters. However, any curve has a 4-parameter family of reparametrizations

$(u, v) \mapsto (au + bv, cu + dv)$, so there are actually a zero dimensional, or finite number, of curves of the general $F = 0$.

The conjecture is known to be true for $d \leq 7$ [16]. For any d, it can even be proven that there exists an infinitesimally rigid curve of degree d on a general X. Similar conjectures can be stated for other Calabi-Yau threefolds.

There are many kinds of non-rational curves which appear to occur in finite number on a general quintic threefold. For instance, elliptic cubic curves are all planar. The plane P that one spans meets the quintic in a quintic curve containing the cubic curve. The other component must be a conic curve. This sets up a 1-1 correspondence between elliptic cubics and conics on any quintic threefold. Hence the number of elliptic cubics on a general quintic must be the same as the number of conics, 609250 [16]. Finiteness of elliptic quartic curves has been proven by Vainsencher [23]; the actual number has not yet been computed.

On the other hand, there are infinitely many plane quartics on *any* quintic threefold: take any line in the quintic, and each of the infinitely many planes containing the line must meet the quintic in the original line union a quartic.

If a curve has $N \cong \mathcal{O} \oplus \mathcal{O}(-2)$, then $C \subset X$ deforms to first order. In fact, since $H^0(N)$ is one-dimensional, there is a family of curves on X parametrized by a single variable t, subject to the constraint $t^2 = 0$. In other words, start with a rational curve given parametrically by forms $\alpha_0, \ldots, \alpha_4$, homogeneous of degree d in u and v. Take a perturbation $\alpha_i(u, v; t) = \alpha_i(u, v) + t\alpha_i'(u, v)$, still homogeneous in (u, v). Form the equation $F(\alpha_0, \ldots a_4) = 0$ and formally set $t^2 = 0$; the resulting equation has a 5 dimensional space of solutions for the α_i', which translates into a unique solution up to multiples and reparametrizations of $\mathbb{P}^1$. The curve C but may or may not deform to second order. C deforms to n^{th} order for all n if and only if C moves in a 1-parameter family. A pretty description of the general situation is given in [21].

If C deforms to n^{th} order, but not to $(n+1)^{\text{th}}$ order, then one sees that while C is an isolated point in the moduli space of curves on X, it more naturally is viewed as the solution to the equation $t^{n+1} = 0$ in one variable t. So C should be viewed as a rational curve on X with multiplicity $n + 1$.

If a curve has $N \cong \mathcal{O}(1) \oplus \mathcal{O}(-3)$, C has a 2 parameter space of infinitesimal deformations, and the structure of $\mathcal{M}$ at C is correspondingly more complicated. An example is given in the next section. The general situation has not yet been worked out.

2. Counting Rational Curves

In this section, a general procedure for calculating the number of smooth rational curves of a given type is described. Alternatively, a canonical definition of this number can be given using the Hilbert scheme (this is what was done by Ellingsrud and Strømme in their work on twisted cubics [11]); however, it is usually quite difficult to implement a calculation along these lines.

Embed the Calabi-Yau threefold X in a larger compact space $\mathbb{P}$ (which may be thought of as a projective space, a weighted projective space, or a product of such spaces). $\mathcal{M}_\lambda$ will denote the moduli space parametrizing smooth rational curves in $\mathbb{P}$ of a given topological type or degree λ. $\mathrm{Def}(X)$ denotes the irreducible component of X in the moduli space of Calabi-Yau manifolds in $\mathbb{P}$ (here the Kähler structure is ignored). In other words, $\mathrm{Def}(X)$ parametrizes the deformations of X in $\mathbb{P}$.

1. Find a compact moduli space $\bar{\mathcal{M}}_\lambda$ containing $\mathcal{M}_\lambda$ as a dense open subset, such that the points of $\bar{\mathcal{M}}_\lambda - \mathcal{M}_\lambda$ correspond to degenerate curves of type λ. $\bar{\mathcal{M}}_\lambda$ parametrizes degenerate deformations of the smooth curve (not the mapping from $\mathbb{CP}^1$ to X). It is better for $\bar{\mathcal{M}}_\lambda$ to be smooth.

2. Find a rank $r = \dim(\mathcal{M}_\lambda)$ vector bundle $\mathcal{B}$ on $\bar{\mathcal{M}}_\lambda$ such that
   (a) To each $X' \in \mathrm{Def}(X)$ there is a section $s_{X'}$ of $\mathcal{B}$ which vanishes at $C \in \bar{\mathcal{M}}_\lambda$ if and only if $C \subset X'$.
   (b) There exists an $X' \in \mathrm{Def}(X)$ such that $s_{X'}(C) = 0$ if and only if $C \in \mathcal{M}_\lambda$ and $C \subset X'$.
   (c) C is an isolated zero of $s_{X'}$ with $\mathrm{mult}_C(s_{X'}) = 1$ if and only if $N_{C/X} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

Working Definition. The *number of smooth rational curves n_λ of type λ* is given by the rth Chern class $c_r(\mathcal{B})$.

Why is this a reasonable definition? Suppose that an $X' \subset \mathrm{Def}(X)$ can be found with the properties required above, with the additional property that there are only finitely many curves of type λ on X', and that they are all infinitesimally rigid. Then it can be checked that the number of (possibly degenerate) curves of type λ on X' is independent of the choice of X' satisfying the above properties, and is also equal to $c_r(\mathcal{B})$. This last follows since $c_{\mathrm{rank}(E)}(E)$ always gives the homology class of the zero locus Z of any section of any bundle E on any variety Y, whenever $\dim(Z) = \dim(Y) - \mathrm{rank}(E)$. In our case, $0 = \dim(\{\mathrm{lines}\}) = \dim(\mathcal{M}_\lambda) - \mathrm{rank}(\mathcal{B})$. In essentially all known cases, the number of curves has been worked out by the method of this working definition. Examples are given below.

There is a potential problem with this working definition. For families of Calabi-Yau threefolds such that no threefold in the family contains finitely many curves of given type, it may be that the "definition" depends on the choice of compactification and/or vector bundle, i.e. this is not well-defined. My reason for almost calling this method a definition is that it *does* give a finite number corresponding to an infinite family of curves, which *is* well-defined in the case that the Calabi-Yau threefold in question belongs to a family containing some other Calabi-Yau threefold with only finitely many rational curves of the type under consideration.

In the case where the general X contains irreducible singular curves which are the images of maps from $\mathbb{P}^1$, a separate but similar procedure must be implemented to calculate these, since they give rise to instanton corrections as well. For example, there is a 6-parameter family of two-planes in $\mathbb{P}^4$. A two plane P meets a quintic threefold X in a plane quintic curve. For general P and X, this curve is a smooth genus 6 curve. But if the curve acquired 6 nodes, the curve would be rational. This being 6 conditions on a 6 parameter family, one expects that a general X would contain finitely many 6-nodal rational plane quintic

curves, and it can be verified that this is indeed the case. The problem is that there is no way to deform a smooth rational curve to such a singular curve — the dimension of $H^1(\mathcal{O}_C)$ is zero for a smooth rational curve C, but positive for such singular curves, and this dimension is a deformation invariant [14, Theorem 9.9]. For example, if one tried to deform a smooth twisted cubic curve to a singular cubic plane curve by projecting onto a plane, there would result an "embedded point" at the singularity, creating a sort of discontinuity in the deformation process [14, Ex. 9.8.4].

Examples:

1. Let $X \subset \mathbb{P}^4$ be a quintic threefold. Take $\lambda = 1$, so that we are counting lines. Here $\mathcal{M}_1 = G(1,4)$ is the Grassmannian of lines in $\mathbb{P}^4$ and is already compact, so take $\bar{\mathcal{M}}_1 = \mathcal{M}_1$. Let $\mathcal{B} = \mathrm{Sym}^5(U^*)$, where U, the universal bundle, is the rank 2 bundle on $\mathcal{M}_1$ whose fiber over a line L is the 2-dimensional subspace $V \subset \mathbb{C}^5$ yielding $L \subset \mathbb{P}^4$ after projectivization. Note that $\mathrm{rank}(\mathcal{B}) = \dim(\mathcal{M}_1) = 6$. $\mathrm{Def}(X) \subset \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^4}(5)))$ is the subset of smooth quintics. A quintic X induces a section s_X of $\mathcal{B}$, since an equation for X is a quintic form on $\mathbb{C}^5$, hence induces a quintic form on V for $V \subset \mathbb{C}^5$ corresponding to L. Clearly $s_X(L) = 0$ if and only if $L \subset X$. The above conditions are easily seen to hold. $c_6(\mathcal{B}) = 2875$ is the number of lines on X. This calculation is essentially the same as that done for cubics in [2, Thm. 1.3], where dual notation is used, so that the U^* used here becomes the universal quotient bundle Q in [2]. The number 2875 agrees with the result of Candelas et. al. [5].

2. Continuing with the quintic, take $\lambda = 2$. Any conic C necessarily spans a unique 2-plane containing C. Let $G = G(2,4)$ be the Grassmannian of 2-planes in $\mathbb{P}^4$, and let U be the rank 3 universal bundle on G. Put $\bar{\mathcal{M}}_2 = \mathbb{P}(\mathrm{Sym}^2(U^*))$ be the projective bundle over G whose fiber over a plane P is the projective space of conics in P. Clearly $\mathcal{M}_2 \subset \bar{\mathcal{M}}_2$ (but they are not equal; $\bar{\mathcal{M}}_2$ also contains the union of any two lines or a double line in any plane). Let $\mathcal{B} = \mathrm{Sym}^5(U^*)/(\mathrm{Sym}^3(U^*) \otimes \mathcal{O}_{\mathbb{P}}(-1))$ be the bundle on $\bar{\mathcal{M}}_2$ of quintic forms on the 3 dimensional vector space $V \subset \mathbb{C}^5$, modulo those which factor as any cubic times the given conic. $(\mathcal{O}_{\mathbb{P}}(-1)$ is the line bundle whose fiber over a conic is the one dimensional vector space of equations for the conic within its supporting plane. The quotient is relative to the natural embedding $\mathrm{Sym}^3(U^*) \otimes \mathcal{O}_{\mathbb{P}}(-1) \to \mathrm{Sym}^5(U^*)$ induced by multiplication.) $\mathrm{rank}(\mathcal{B}) =$

$\dim(\bar{\mathcal{M}}_2) = 11$. $c_{11}(\mathcal{B}) = 609250$. See [16] for more details, or [9] for an analogous computation in the case of quartics. The number 609250 agrees with the result of Candelas et. al.

3. Again consider the quintic, this time with $l = 3$. In [11], Ellingsrud and Strømme take $\bar{\mathcal{M}}_3$ to be the closure of the locus of smooth twisted cubics in $\mathbb{P}^4$ inside the Hilbert scheme. This space has dimension 16. $\mathcal{B}$ is essentially the rank 16 bundle of degree 15 forms on $\mathbb{P}^1$ induced from quintic polynomials in P^4 by the the degree 3 parametrization of the cubic (it must be shown that this makes sense for degenerate twisted cubics as well). The equation of a general quintic gives a section of $\mathcal{B}$, vanishing precisely on the set of cubics contained in X. $c_{16}(\mathcal{B}) = 317206375$, again agreeing with the result of Candelas et. al.

The key to the first two calculations are the Schubert calculus for calculating in Grassmannians [13, Ch. 1.5] and standard formulas for projective bundles [14, Appendix A.3]. The third calculation is more intricate.

Regarding complete intersection Calabi-Yau manifolds, similar examples are found in [17] for lines and [22] for conics.

Note that the number of curves $c_r(\mathcal{B})$ in no way depends on the choice of X, even if X is a degenerate Calabi-Yau threefold, or contains infinitely many rational curves of type λ. It turns out that a natural meaning can be assigned to this number. In fact, the moduli space of curves of type λ on X splits up into "distinguished varieties" Z_i [12], and a number, the *equivalence* of Z_i, can be assigned to each distinguished variety (the number is 1 for each $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ curve). This number is precisely equal to the number of curves on X in Z_i which arise as limits of curves on X' as X' approaches X in a 1-parameter family [12, Ch. 11].

Examples:

1. If a quintic threefold is a union of a hyperplane and a quartic, then the quintic contains infinitely many lines and conics. However, in the case of lines, given a general 1-parameter family of quintics approaching this reducible quintic, 1275 lines approach the hyperplane, and 1600 lines approach the quartic [15]. So the infinite set of lines in the hyperplane "count" as 1275, while those in the quartic count as 1600. For counting conics, the 609250 conics distribute themselves as 187250 corresponding to the component of conics in the hyperplane, 258200 corresponding to the

component of conics in the quartic, and 163200 corresponding to the component of conics which degenerate into a line in the hyperplane union an intersecting line in the quartic [18]. Note that this reducible conic lies in $\bar{\mathcal{M}}_2 - \mathcal{M}_2$, and illustrates why $\mathcal{M}_2$ itself is insufficient for calculating numbers when there are infinitely many curves. Most of these numbers have been calculated recently by Xian Wu [24] using a different method.

2. If a quintic threefold is a union of a quadric and a cubic, the lines on the quadric count as 1300, and the lines on the cubic count as 1575 [15, 25]. The conics on the quadric count as $215,950$, while the conics on the cubics count as 243900 [25]. Presumably this implies that the conics which degenerate into a line in the quadric union an intersecting line in the cubic count as $609250 - (215950 + 243900) = 149400$, but this has not been checked directly yet.

3. There are infinitely many lines on the Fermat quintic threefold $x_0^5 + \ldots + x_4^5 = 0$. These divide up into 50 cones, a typical one being the family of lines given parametrically in the homogeneous coordinates (u, v) of $\mathbb{P}^1$ by $(u, -u, av, bv, cv)$, where (a, b, c) satisfy $a^5 + b^5 + c^5 = 0$. Each of these count as 20. There are also 375 special lines, a typical one being given by the equations $x_0 + x_1 = x_2 + x_3 = x_4 = 0$ (these lines were also noticed in [10]). These lines L count with multiplicity 5. Note that $50 \cdot 20 + 375 \cdot 5 = 2875$ [1]. This example illustrates the potential complexity in calculating the distinguished varieties Z_i — some components can be embedded inside others. This may be understood as well by looking at the moduli space of lines on the Fermat quintic locally at a line corresponding to a special line. Since $N_{L/X} \cong \mathcal{O}(1) \oplus \mathcal{O}(-3)$, $H^0(N_{L/X})$ has dimension 2, and the moduli space of lines is a subset of a 2-dimensional space. A calculation shows that it is locally defined inside 2 dimensional (x, y) space by the equations $x^2 y^3 = x^3 y^2 = 0$. The x and y axes correspond to lines on each of the 2 cones, each occurring with multiplicity 2; there would be just one equation $x^2 y^2 = 0$ if the special line corresponding to $(0, 0)$ played no role; since this is not the case, it can be expected to have its own contribution; i.e. each special line is a distinguished variety.

4. Examples for lines in complete intersection Calabi-Yau threefolds were worked out in [17].

These numbers can also be calculated by intersection theory techniques. Let $s(Z_i, \bar{\mathcal{M}}_\lambda)$ denote the *Segre class* of Z_i in $\bar{\mathcal{M}}_\lambda$. If Z_i is smooth, this is simply the formal inverse of the total Chern class

$1 + c_1(N) + c_2(N) + \ldots$ of the normal bundle N of Z_i in $\bar{\mathcal{M}}_\lambda$. Then if Z_i is a connected component of the zero locus of s_X, the equivalence of Z_i is the zero dimensional part of $c(\mathcal{B}) \cap s(Z_i, \bar{\mathcal{M}}_\lambda)$ [12, Prop. 9.1.1].

If Z_i is an irreducible component which is not a connected component, then this formula is no longer applicable. However, I have had recent success with a new method that supplies "correction terms" to this formula. The method is currently ad hoc (the most relevant success I have had is in calculating the "number" of lines on a cubic surface which is a union of three planes), but I expect that a more systematic procedure can be developed.

This of course is a reflection of the situation in calculating instanton corrections to the Yukawa couplings. If there is a continuous family of instantons, then calculating the corrections will be more difficult. If the parameter space for instantons is smooth, this should make the difficulties more managable. If the space is singular, the calculation is more difficult. I expect that the calculation would be even more difficult if instantons occur in at least 2 families that intersect.

Of course, in the calculation of the Yukawa couplings via path integrals, there is no mention of vector bundles on the moduli space. This indicates to me that there should be a mathematical definition of the equivalence of a distinguished variety that does not refer to an auxiliary bundle. Along these lines, one theorem will be stated without proof.

Let Z be a k-dimensional unobstructed family of rational curves on a Calabi-Yau threefold X. There is the total space $\mathcal{Z} \subset Z \times X$ of the family, with projection map $\pi : \mathcal{Z} \to Z$ such that $\pi^{-1}(z)$ is the curve in X corresponding to z, for each $z \in Z$. Let N be the normal bundle of $\mathcal{Z}$ in $Z \times X$. Define the equivalence $e(Z)$ of Z to be the number $c_k(R^1\pi_* N)$. For example, if $k = 0$, then the curve is infinitesimally rigid, and $e(Z) = 1$.

Theorem. *Let Z be an unobstructed family of rational curves of type λ on a Calabi-Yau threefold X. Suppose that X deforms to a Calabi-Yau threefold containing only finitely many curves of type λ. Then precisely $e(Z)$ of these curves (including multiplicity) approach curves of Z as the Calabi-Yau deforms to X.*

Some of the examples given earlier in this section can be redone via this theorem. Also, as anticipated by [5], it can be calculated that a factor of $1/m^3$ is introduced by degree m covers by a calculation similar in spirit to that found in [3].

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